Math 279 Lecture 10 Notes

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1 Kolmogorov's Continuity Theorem for Rough Paths and Candidates for the Lift of Brownian Motion

1.1 Kolmogorov's continuity theorem for rough paths

Recall that if $A(x) = \int_0^T \int_0^T \psi(\frac{|x(t)-x(s)|}{p(|t-s|)})\, dt\, ds$ with $\psi, p : [0,\infty) \to [0,\infty)$ increasing, $\psi(0) = p(0) = 0$ and $\psi(\infty) = \infty$, then

$$|x(t) - x(s)| \leq 8 \int_0^{|t-s|} \psi^{-1}\left(\frac{4A}{\theta^2}\right) p(d\theta).$$

For example, if $\psi(r) = r^q$ and $p(r) = r^{\alpha+1/q}$ with $q > 1$ and $\alpha > 0$, then

$$|x(t) - x(s)| \leq c_0(q,\alpha) A(x)^{1/q} |t-s|^{\alpha - 1/q}.$$

In summary, if

$$A(x) = \int_0^T \int_0^T \frac{|x(t) - x(s)|^q}{|t-s|^{\alpha q + 1}}\, dt\, ds,$$

then x is Hölder continuous of exponent $\alpha - 1/q$. In particular, if x is randomly selected according to a probability measure $\mathbb{P}$ and $\mathbb{E}[|x(t) - x(s)|^q] \leq c_0 |t-s|^{\beta q}$, then

$$\mathbb{E}[A(x)] \leq c_0 \int_0^T \int_0^T |t-s|^{\beta q - \alpha q - 1}\, dt\, ds < \infty$$

if $\beta > \alpha$. In summary, if we have this L^q bound on $x(t) - x(s)$, then x is Hölder of exponent $\gamma \in (0, \beta - 1/q)$. This is also true for $x : [0,T]^d \to \mathbb{R}^\ell$: If $(\mathbb{E}[|x(t) - x(s)|^q])^{1/q} \leq c_0 |t-s|^\beta$, then x is Hölder of exponent $\gamma \in (0, \beta - d/q)$.

Here is a version of Kolmogorov's continuity theorem that involves rough paths:

Theorem 1.1. *Let $x : [0,T] \to \mathbb{R}^\ell$ and its lift $\mathbb{X} : [0,T]^2 \to \mathbb{R}^{\ell \times \ell}$ satisfy Chen's relation:*

$$\mathbb{X}(s,t) = \mathbb{X}(s,u) + \mathbb{X}(u,t) + x(s,u) \otimes x(u,t).$$

Let $q \geq 2$, $\beta > 1/q$, and assume that there exists a constant c_0 such that $(\mathbb{E}[|x(s,t)|^q])^{1/q} \leq c_0|t-s|^\beta$ and $(\mathbb{E}[(\sqrt{|\mathbb{X}(s,t)|})^q])^{1/q} \leq c_0|t-s|^\beta$. Then there is a version of $\mathbf{x} = (x, \mathbb{X})$ such that

$$\mathbb{E}\left[\left(\sup_{s\neq t} \frac{x(s,t)}{|t-s|^{\alpha-1/q}}\right)^q + \left(\sup_{s\neq t} \frac{\sqrt{|\mathbb{X}(s,t)|}}{|t-s|^{\alpha-1/q}}\right)^q\right] < \infty,$$

provided that $\alpha < \beta$.

Proof. Without loss of generality, assume $T = 1$. Take a dyadic approximation of $[0,1]$: set $D_n = \{j/2^n : 0, 1, \ldots, 2^n\}$, and let $D = \bigcup_{n=1}^\infty D_n$, which is dense in $[0,1]$. Set

$$A_n = \sup_{t\in D_n} |x(t+2^{-n}) - x(t)| = \sup_{t\in D_n} |x(t, t+2^{-n})|, \qquad B_n = \sup_{t\in D_n} |\mathbb{X}(t, t+2^{-n})|$$

Let $s, t \in D$ with $s < t$, and pick m so that $1/2^{m+1} < |s-t| \leq 1/2^m$. Pick $\theta \in [s,t] \cap D_m$, which exists because $|s-t| \geq 1/2^m$. Then

$$|x(t) - x(s)| \leq |x(t) - x(\theta)| + |x(\theta) - x(s)|.$$

Now write the dyadic expansion $t - \theta = \frac{a_0}{2^m} + \frac{a_1}{2^{m+1}} + \cdots$, so $|x(t) - x(\theta)| \leq \sum_{n\geq m} A_n$. Doing the same with the second term,

$$\leq 2\sum_{n\geq m} A_n$$

Hence,

$$\begin{aligned}\frac{|x(t)-x(s)|}{|t-s|^\gamma} &\leq |x(t)-x(s)|2^{(m+1)\gamma} \\ &\leq 2^{\gamma+1}\sum_{n\geq m} A_n 2^{m\gamma} \\ &\leq 2^{\gamma+1}\sum_{n\geq m} A_n 2^{n\gamma}.\end{aligned}$$

So we get the bound

$$\sup \frac{|x(t)-x(s)|}{|t-s|^\gamma} \leq 2^{\gamma+1}\sum_{n=0}^\infty A_n 2^{n\gamma}.$$

We want to get a bound on the L^q norm of this:

$$\left(\mathbb{E}\left[\left(\sup \frac{|x(t)-x(s)|}{|t-s|^\gamma}\right)^q\right]\right)^{1/q} \leq 2^{\gamma+1}\sum_n (\mathbb{E}[A_n^q])^{1/q} 2^{n\gamma}.$$

On the other hand,

$$A_n^q = \sup_{t\in D_n} |x(t+2^{-n}) - x(t)|^q \leq \sum_{t\in D_n} |x(t+2^{-n}) - x(t)|^q,$$

and taking expectations gives

$$\begin{aligned}\mathbb{E}[A_n^q] &\leq \sum_{t\in D_n}\mathbb{E}[|x(t+2^{-n})-x(t)|^q]\\ &\leq c_0^q 2^n 2^{-n\beta q}.\end{aligned}$$

This gives the L^q norm bound

$$(\mathbb{E}[A_n^q])^{1/q} \leq c_0 2^{-n(\beta-1/q)}.$$

Hence,

$$\left(\mathbb{E}\left[\left(\sup\frac{|x(t)-x(s)|}{|t-s|^\gamma}\right)^q\right]\right)^{1/q} \leq c_0 2^{\gamma+1}\sum_n 2^{-n(\beta-1/q-\gamma)} < \infty$$

if $\gamma < \beta - 1/q$.

As for $\mathbb{X}(s,t)$, we do likewise. Let s, t, θ be as above and use

$$\mathbb{X}(s,t) = \mathbb{X}(s,\theta) + \mathbb{X}(\theta,t) + x(s,\theta)\otimes x(\theta,t).$$

We get

$$|\mathbb{X}(s,t)| \leq 2^{\gamma+1}\sum_n B_n 2^{n\gamma} + \left(\sum_n A_n e^{n\gamma}\right)^2,$$

and we can repeat the above argument.

This would give us the regularity of x (resp. $\mathbb{X}$) restricted to D (resp. D^2). Then set $\widetilde{x}(t) = \lim_{\substack{t_n\to t\\ t_n\in D}} x(t_n)$, and we can show that $x = \widetilde{x}$ almost surely:

$$\begin{aligned}\mathbb{E}[|x(t)-\widetilde{x}(t)|] &= \mathbb{E}\left[\lim_{n\to\infty}|x(t)-x(t_n)|\right]\\ &\leq \underbrace{\liminf \mathbb{E}[|x(t)-x(t_n)|]}_{\leq c_0|t-t_n|^\beta}\\ &= 0.\end{aligned}$$

□

1.2 Candidates for the lift of Brownian motion

We now offer two candidates for the lift of an ℓ-dimensional Brownian motion, namely Itô and Stratanovich. Define

$$\mathbb{B}^{\text{Itô}}(s,t) = A(s,t) - B(s)(B(t)-B(s)),$$

with

$$A(s,t) = \lim_{n\to\infty}\sum_{t_i \text{ dyadic in } [s,t]} B(t_i)(B(t_{i+1})-B(t)).$$

Define the Stratanovich integral similarly except with

$$A^{\text{Strat}}(s,t) = \lim_{n\to\infty} \sum_{t_i \text{ dyadic in } [s,t]} \frac{B(t_i)+B(t_{i+1})}{2}(B(t_{i+1})-B(t)).$$

For the sake of definiteness, assume $s=0$. For diagonal terms, we have

$$A^{\text{Itô}}_{r,r} = \lim_{n\to\infty} \sum_{\{t_i\}=D_n} B_r(t_i)(B_r(t_{i+1})-B_r(t_i)),$$

$$A^{\text{Strat}}_{r,r} = \lim_{n\to\infty} \sum_{\{t_i\}=D_n} \frac{B_r(t_i)+B_r(t_{i+1})}{2}(B_r(t_{i+1})-B_r(t_i)) = \frac{B(t)^2-B(s)^2}{2}.$$

Observe that

$$(A^{\text{Strat}}_{r,r} - A^{\text{Itô}}_{r,r})(s,t) = \lim \sum_i \frac{1}{2}(B_r(t_{i+1})-B_r(t_i))^2 = \frac{t-s}{2},$$

where the last step is a theorem of Lévy. (The proof is to show that $\mathbb{E}[\sum(B_r(t_{i+1}) - B_r(t_i))^2 - (t_{i+1}-t_i)]^2 \to 0$ as $n\to\infty$.) Hence,

$$A^{\text{Itô}}_{r,r}(s,t) = \frac{B(t)^2-B(s)^2-(t-s)}{2}.$$

It remains to evaluate $A_{r,r'}/A^{\text{Strat}}_{r,r'}$. Basically, we have 2 independent, one dimensional standard Brownian motions, say B and B', and we want to calculate $\lim \sum_i B'(t_i)(B(t_{i+1}) - B(t_i))$. Let

$$\mathbb{B}_n(t) = \sum_{i=0}^{\lfloor t2^n \rfloor -1} B'(t_i)B(t_i,t_{i+1}).$$

First assume $t=1$, and let us examine

$$\mathbb{B}_{n+1} - \mathbb{B}_n = \sum_i (B'(t_i)B(t_i s_i) + B'(s_i)B(s_i,t_{i+1}) - B'(t_i)B(t_i,t_{i+1})),$$

where s_i is the midpoint of $[t_i,t_{i+1}]$.

$$= \sum_i B'(t_i,s_i)B(s_i,t_{i+1}).$$

So

$$\mathbb{E}[(\mathbb{B}_{n+1}-\mathbb{B}_n)^2] = \sum_i \mathbb{E}[B'(t_i,s_i)^2 B(s_i,t_{i+1})^2]$$

$$
\begin{aligned}
&= \sum_i 2^{-2(n+1)} \\
&= 2^{-n-2}.
\end{aligned}
$$

Hence, $\mathbb{B}_n$ is Cauchy in L^2.

It turns out that $\mathbb{B}_n$ as a function of time is a martingale, and we can take advantage of this to have a better convergence. First, we set $\mathcal{F}_t$ to be the σ-algebra generated by $(B(s) : s \in [0,t])$, and we say $t \mapsto M(t)$ is a martingale if $\mathbb{E}[M(t) \mid \mathcal{F}(s)] = M(s)$ for $s < t$. Using Doob's inequality, we can have convergence that is uniform in t:

$$
\left(\mathbb{E}\left[\left| \sup_{t\in[0,T]} M(t) \right|^p \right] \right)^{1/p} \le \frac{p}{p-1}\, \mathbb{E}[|M(T)|^p], \qquad p > 1.
$$